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Nonlinear Analysis of Leaf Springs of Functionally Graded Materials

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Abstract

The present study deals with the numerical analysis of large deflection of prismatic cantilever beams for various types of material properties with a transverse load at free end, to study the displacement response of leaf springs. Besides the free end displacement, the variation of stress, strain and the bending moment of the beam having variable material properties with the beam length are obtained by the technique of minimization of total potential energy. The mathematical formulation is based on a variational principle using Galerkin's assumed mode method. The displacement functions are approximated by linear combination of sets of orthogonal coordinate functions, developed through Gram-Schmidt scheme and substituted in the governing equilibrium equation. The final solution of the large displacement geometric nonlinear problem is obtained iteratively with the help of MATLAB computational simulation. It is observed that the free end displacements and the shortening of projected beam length are greatly affected by the variation in elasticity modulus value. The present computational method has been validated and some new results have been furnished. The influence of material gradation for various types of exponential and parabolic distribution is shown for three different types of loading.

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Keywords: Material Nonlinearity, Leaf Spring, Cantilever Beam, Variable Material Properties

Nomenclature

l	current horizontal length of beam
p	magnitude of uniform distributed load
w	transverse displacement field
C_i	unknown coefficients
E_0	elastic modulus of beam material
I	area moment of beam section
L	length of beam
P	magnitude of concentrated load
U, V	strain energy stored in the system and potential energy of external forces
Greek symbols	
δ	variational operator, beam deflection at free end
ϕ_i	Set of orthogonal functions
ξ	Normalized axial coordinate

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1. Introduction

Traditionally leaf springs are modeled as a bundle of prismatic cantilever beams with gradually reducing lengths and analysis is carried out with the small displacement theory. For better characterization of such beams geometric non-linear analysis is carried out and moreover updated Lagrangian analysis considers the changed geometry of beam as well. However the deflection characteristics of leaf springs are highly nonlinear and those modeling aims to find the optimized shapes of the leaf spring. Recent development in functionally graded materials (FGM) offers spatial variation of physical properties through non-uniform distribution of two different materials microstructures.

Deflections and stresses for non-linear bending are discussed and compared with those of a traditional leaf spring by Rajendran and Vijayarangan [1]. Design and manufacture of automotive leaf spring using functionally graded and composite materials have been addressed by Shokrieh [2] and Qureshi [3]. Sugiyama et al [4] reported development of nonlinear elastic leaf spring model for multi body vehicle systems. Rahman et al [5] carried out non-linear geometric analysis of parabolic leaf spring. Almeida et al [6] used a tailored Lagrangian formulation for functionally graded cantilever beams of rectangular and hollow circular cross-section. Simsek [7] proposed non-linear vibration analysis of a functionally graded Timoshenko beam under action of a moving harmonic load, in which material properties of the beam vary continuously in thickness direction according to a power-law form. Wattanasakulpong et al [8] analyzed free vibration analysis of layered functionally graded beams with experimental validation, and employed third order shear deformation theory to formulate a governing equation for predicting free vibration of layered functionally graded beams.

In this paper, we investigate numerically large deformation of leaf spring systems, considering the effect of nonlinear material property variations. The proposed method is validated with Ref. [5] for nonlinear stress and deformation analysis of a parabolic leaf spring. The present study undertakes a prismatic cantilever beam with geometric nonlinearity due to large deflection and reports beam characteristic curves in normalized load-deflection plane.

Mathematical formulation

The numerical solution of the governing differential equation of the beam system in question is derived by using minimization of total potential energy, based on the Bernoulli-Euler theory. Mathematically the variational principle is represented as $\delta(\pi)=0$, where δ is the variational operator and π is the total potential energy. Noting that $\pi = U + V$, the equation can be expressed as, $\delta(U+V)=0$. The strain energy of the beam is, $U=\frac{1}{2}\int_0^1 E_x\left(d^2w/dx^2\right)^2 dx$ and the expression of the potential energy is, $V=-Pw\,|_{x=L}-\int_0^L (pw)dx$. The potential energy comes from the work done by the external uniform (p) and concentrated (P) load at the tip of the cantilever beam. Substituting the expressions of U and V in equation $\delta(\pi)=0$, and after carrying out some mathematical manipulations, we get the system governing equation in the normalized length coordinate $\xi(=x/L)$, as furnished below.

$$\delta\left[\frac{1}{2L^3}\int_0^1 E_\xi\left(d^2w/dx^2\right)^2 d\xi - Pw\,|_{\xi=1} - L\int_0^1 pw\,d\xi\right]=0. \tag{1}$$

It should be noted that E_ξ is taken within the integration limit to accommodate variation in material property of a FGM beam. In Eq. (1), the displacement functions $w(\xi)$ can be approximated by a linear combination of sets of orthogonal coordinate functions as $w(\xi)\cong\sum c_i\phi_i$, i = 1, 2, ..., n. The set of orthogonal functions ϕ_i are developed through Gram-Schmidt scheme, in which a starting function is used to generate the higher order orthogonal functions. The starting function ϕ_1 necessary in the first hand, is selected by satisfying the boundary conditions at fixed and free end as given below.

$w\,|_{x=0}=0$, $\frac{dw}{dx}\,|_{x=0}=0$, at fixed end and $E_x I\frac{d^2w}{dx^2}\,|_{x=L}=0$, $E_x I\frac{d^3w}{dx^3}\,|_{x=L}=0$ at free end.

Substitution of $w(\xi)\cong\sum c_i\phi_i$ in Eq. (1) yields the solution for the unknown coefficients c_i and subsequently displacement field w is obtained. The detail description of the mathematical procedure is omitted here to maintain brevity.

Moreover, detail derivations of the equations used in this paper is available in [9], in connection with the analysis of a non-uniform leaf spring of homogeneous linear elastic material behaviour. The mathematical formulation as presented in Eq. (1) is applicable for small deflection only $(\delta \cong t/L)$, beyond which strain-displacement relations become non-linear. In the present paper, the effect of geometric nonlinearity is implemented iteratively, in which the total load on the beam is imposed incrementally. In each load step, a correction on projected beam length is carried out such that the length of the deflection curve remains constant to the original straight length of the beam. This is shown through a schematic representation of the bending curve in Fig. 1(a). The elemental beam length shown in figure is given by

$$ds = \left(1 + (dw/dx)^2\right)^{1/2} dx \ . \tag{2}$$

Equation (2) is integrated to obtain the stretching of beam length

$$\Delta = \int_0^L \left[\left\{1 + (dw/dx)^2\right\}^{1/2} - 1\right] dx \ . \tag{3}$$

Δ has horizontal ΔL and vertical $\left(w|_{x=L}\right)$ components which are determined to obtain the effective beam length for the next incremental load step, i.e., $\left(l_i = l_{i-1} - \Delta L\right)$. It should be noted that, in each load step, analysis is carried out following Bernoulli-Euler theory for non-homogeneous beam material. Determination of ΔL requires slope of the beam deflection curve at free end and it is obtained from the slope of the current deflection profile. The process is continued incrementally till the value of normalized load parameter becomes 10. The properties of the beam material is assumed to vary with the length of the beam following the relations, i) $E_\xi = E_0 \exp\left(-n\xi^k\right)$ and ii) $E_\xi = E_0\left(1 - n\xi^k\right)$ respectively, where E_0 is the modulus of elasticity at fixed end. Thus the material property of the beam varies exponentially and parabolically along with its length.

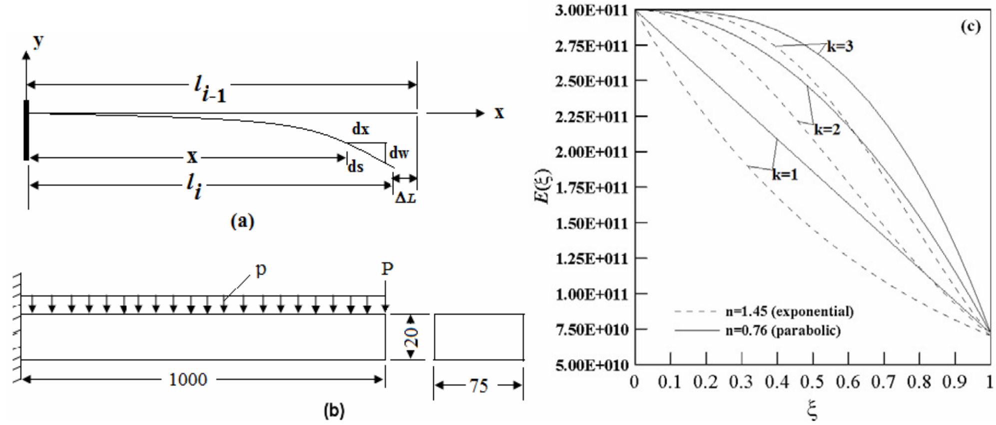


Fig. 1: (a) Bending curve of a cantilever beam under large deformation (b) Geometry of prismatic cantilever beam and (c) Variation of elasticity modulus with length for specific values of parameter *n* and *k*.

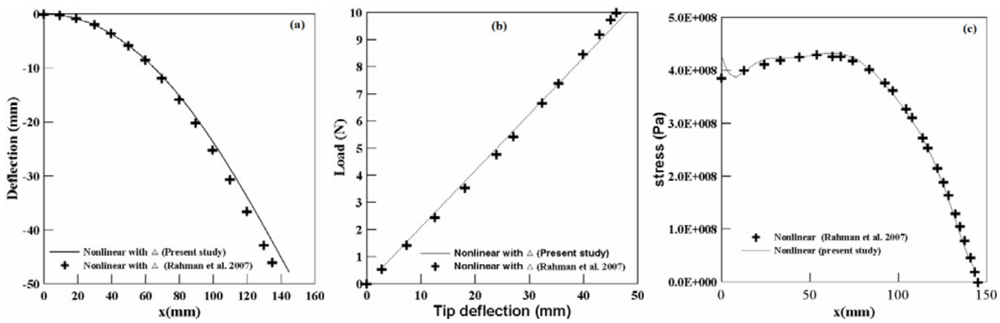
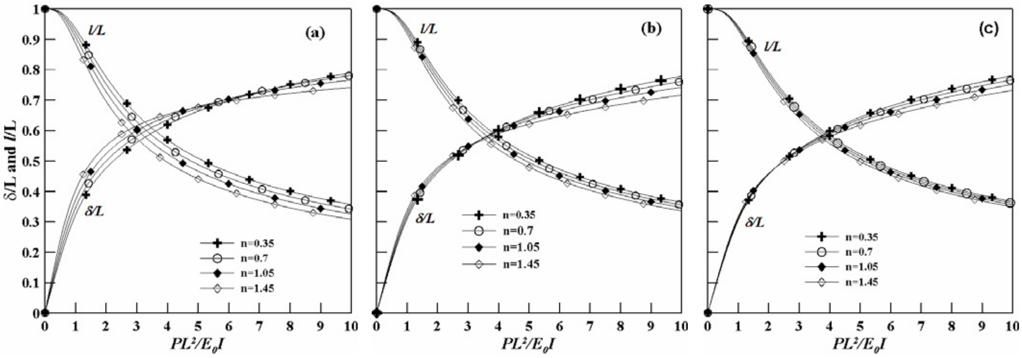
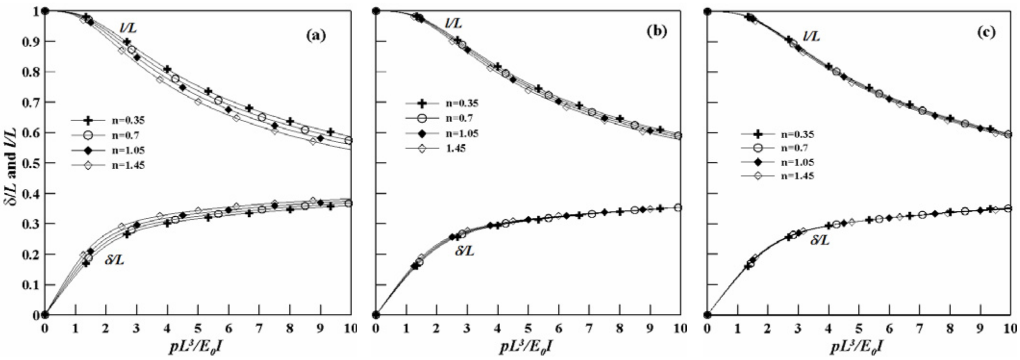


Fig. 2: Validation of present results with Rahman [5] for (a) variation of deflection with beam length x, (b) variation of load with tip deflection and (c) variation of bending stress with beam length

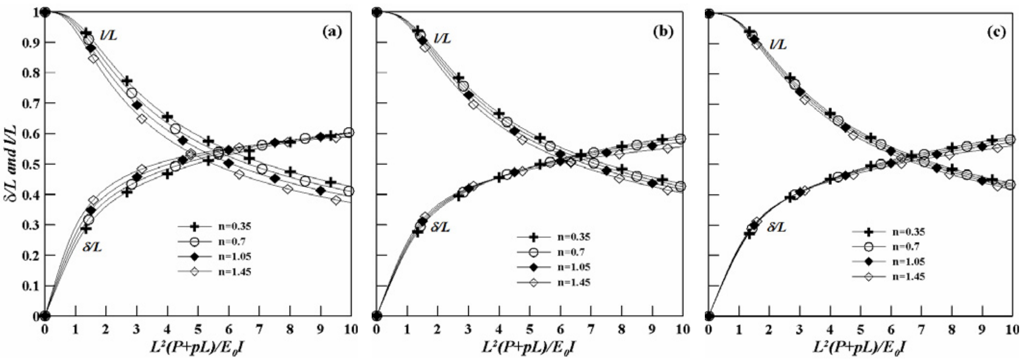
The dimensional details of the beam are shown in Fig. 1(b), where dimensions are in mm. The transverse loads indicated in the figure is combined but they are applied separately also, i.e., as uniform, concentrated and combined, while generating the results. Variation of elasticity modulus with length is shown in Fig. 1(c) for some specific values of parameter n and k. Values of parameter n for exponential and parabolic distribution are so selected that the values of elasticity modulus at fixed (300 GPa) and free (75 GPa) end becomes identical, as indicated in the figure.



i) Cantilever beam under concentrated load (load parameter $PL^2 / E_0 I$)

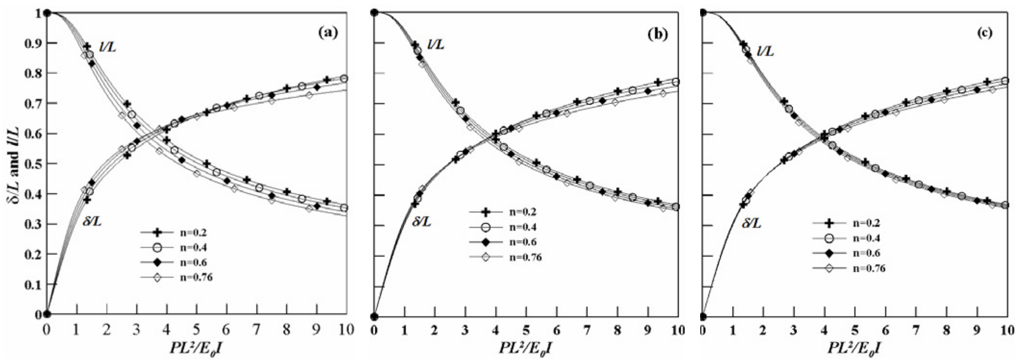


ii) Cantilever beam under uniformly distributed load (load parameter $pL^3 / E_0 I$)

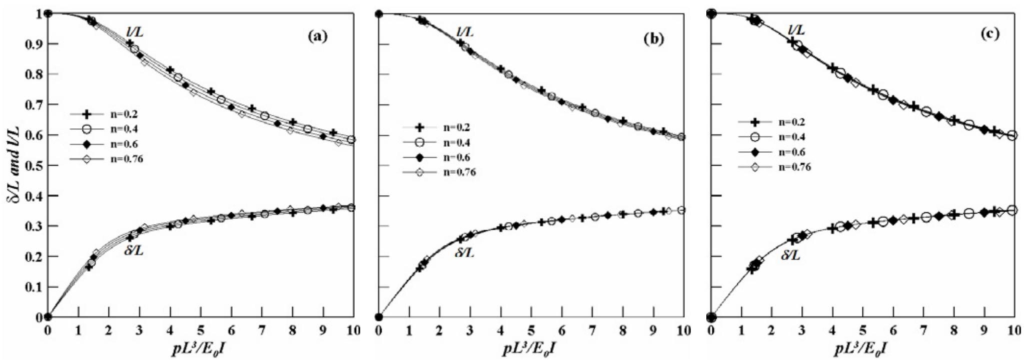


iii) Cantilever beam under combined load (load parameter $L^2(P + pL)E_0 I$)

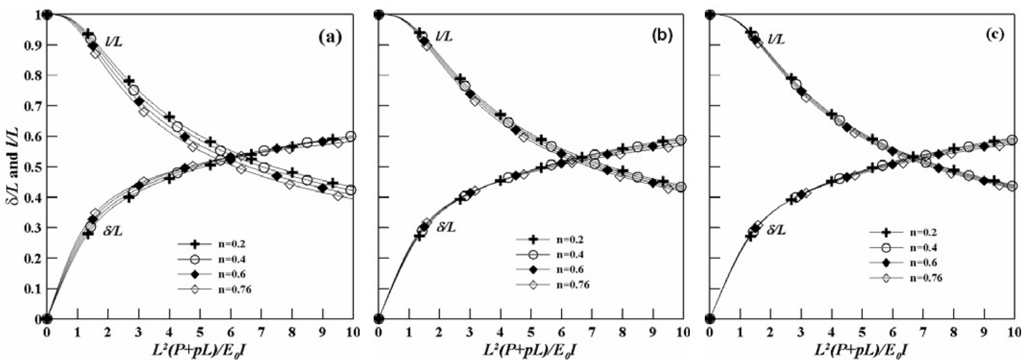
Fig. 3: Variation of δ/L and l/L with load parameter for exponential variation in material property (a) for k=1 (b) for k=2 and (c) for k=3



i) Cantilever beam under concentrated load (load parameter $PL^2 / E_0 I$)



ii) Cantilever beam under uniformly distributed load (load parameter $pL^3 / E_0 I$)



iii) Cantilever beam under combined load (load parameter $L^2(P + pL)E_0 I$)

Fig. 4: Variation of δ/L and l/L with load parameter for parabolic variation in material property (a) for k=1 (b) for k=2 and (c) for k=3

3. Result and Discussion

Validation of present results for a prismatic cantilever beam is carried out with the analytical results of Rahman et al [5] for concentrated load and shown in Fig. 2 (a-c). The comparative results in Fig. 2 correspond to an isotropic beam with parabolic variation in width acted upon by concentrated load, the details of which are omitted here to maintain brevity. The agreements of the results establish the validity of the present method. The present study deals with the analysis of large deflection of cantilever beams for non-homogeneous material properties under transverse load. It should be noted that variation in material property provides higher value of modulus of elasticity towards the root. The reference value of E (E_0) is considered at the fixed-end of the beam and the load parameter is calculated based on E_0.

As mentioned earlier beam deflection is characterized for three different types of loading, namely concentrated, uniformly distributed and combined load. The normalized load parameter used in these cases are PL^2/E_0I, pL^3/E_0I and $L^2(P+pL)E_0I$ respectively and in each case value of normalized load parameter is varied upto 10. This variation is effected incrementally through 100 load steps and corresponding to each value of loading the projected beam length (l) and maximum beam deflection $(w|_{x=L})$ is noted (defined as δ in the figures).

Variation of normalized values of tip deflection (δ/L) and beam shortening (l/L) with load parameter for the concentrated, uniform and combined loads are shown in Figs. 3, 4 for exponential and parabolic variation in elasticity modulus. The normalized beam characteristic curves are furnished for different values of material parameter n and k, and those are indicated in the respective figures. It is seen that concentrated loading produces maximum deflection followed by combined and uniform loading, for all types of variation in material properties. Comparison of the corresponding graphical plots of beam characteristic curves of Fig. 3 and Fig. 4 reveals that effects of exponential and parabolic distributions in material property variation is insignificant. Comparison of figures (a), (b) and (c) for all types of loading show that higher value of geometry parameter k suppresses the effect of parameter n as the curves of different n values come closer together. Observations from individual plots indicate that the magnitude of beam shortening is more for higher values of n but this behavior is not consistent for tip deflection. The observations made in normalized plane do not reflect the actual physical behavior of a beam and as such further study on the effect of material property variation is called for.

4. Conclusion

An energy based variational method is proposed to analyze geometric nonlinearity of uniform leaf springs following the concept of updated Lagrangian analysis. The system is formulated as cantilever beam problem in which the effect of material property variations are considered. The fundamental formulation is based on a variational method using total potential energy functional and solution is sought through Galerkin's assumed mode method. The final solution of the large displacement geometric nonlinear problem is obtained iteratively with the help of MATLAB computaional simulation. The present computational method has been successfully validated with existing results and some new results have been furnished. Due to adequate improvement of mechanical properties it is seen that FGM leaf springs are more economical than conventional leaf springs. The present method, being based on an iterative computational technique, may be used to extend the problem in the area of thermo-elasticity.

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